Turbulent Prandtl Number Within a Near-Wall Flow

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Abstract

THE turbulent Prandtl number variation within the near-wall region of fully developed boundary layers is obtained by using profile analysis. This is achieved by deriving expressions for the "eddy" diffusivities from Spalding's inner layer formula and its thermal analog. The analysis is performed over two orders of magnitude of the molecular Prandtl number, encompassing most of the common, nonmetallic fluids. The turbulent Prandtl number distribution is found to be complex and dependent on its molecular counterpart within the diffusive sublayer. This behavior has important consequences for the treatment of the near-wall region in numerical calculation methods for thin shear layers.

Contents

Profile Analysis

Many and varied turbulence models have been devised for the "Reynolds" shear stress, whereas by far the most common practice for modeling the corresponding heat flux is to prescribe a value for the "turbulent Prandtl number" Pr_t (the ratio of the turbulent or "eddy" diffusivities for the momentum and heat). The latter diffusivities appear in the classical Boussinesq formulations for the total (molecular plus turbulent) shear stress and heat flux, which may be written, using conventional near-wall scaling,[1] in the form

$$\tau^+ = (1+\epsilon_m^+)\,(du^+/dy^+) \tag{1}$$

$$q^+ = \left(\frac{1}{Pr}+\epsilon_\theta^+\right)\frac{dT^+}{dy^+} = \left(\frac{1}{Pr}+\frac{\epsilon_m^+}{Pr_t}\right)\frac{dT^+}{dy^+} \tag{2}$$

as, by definition,

$$Pr_t \equiv \epsilon_m^+/\epsilon_\theta^+ \tag{3}$$

The turbulent Prandtl number was developed by analogy with its molecular counterpart Pr, although it is normally considered to be largely a function of the type of flow rather than the fluid properties. Despite the widespread use of Pr_t in conjunction with numerical calculation methods since the mid-1960s, uncertainty remains concerning its dependence, as has been emphasized in the comprehensive reviews by Reynolds[2] and Launder.[3]

The variation of the mean (time-averaged) velocity profile across a constant-stress inner layer is constrained by the following limiting conditions:

$$u^+ = 0 \text{ and } du^+/dy^+ = 1 \text{ at } y^+ = 0 \tag{4}$$

$$u^+ = (1/\kappa)\,\ln y^+ + B, \text{ for } y^+ > 30\text{-}50 \tag{5}$$

The restrictions of Eq. (4) are imposed by the so-called "no-slip" condition at the wall, while the log-law of Eq. (5) results from simple near-wall scaling arguments at high-turbulence Reynolds numbers. Attempts to develop a single inner layer expression for u^+ in terms of y^+ were unsuccessful in meeting these constraints, until Spalding[4] observed that by inverting the problem a corresponding function for $y^+ = y^+(u^+)$ could be readily obtained. Inspection of the limiting conditions (4) and (5), together with the need to insure agreement with experimental data in the intervening region, led him to propose a continuous analytic function of the form

$$y^+ = u^+ + e^{-A}\left[e^{\kappa u^+} - 1 - \sum_{n=1}^{n=4}(\kappa u^+)^n/n!\right] \tag{6}$$

where $A = \kappa B$. In the constant-stress region, for which $\tau^+ = 1$, the eddy viscosity becomes, via Eq. (1),

$$\epsilon_m^+ = (dy^+/du^+) - 1 \tag{7}$$

Thus, differentiation of the velocity profile expression (6) and substitution into Eq. (7) yields a continuous inner layer turbulent viscosity distribution,

$$\epsilon_m^+ = \kappa e^{-A}\left[e^{\kappa u^+} - 1 - \sum_{n=1}^{n=3}(\kappa u^+)^n/n!\right] \tag{8}$$

The resulting $\epsilon_m^+ - y^+$ profile is shown in the full report,[1] for which it was computed numerically using the Newton-Raphson iterative method to solve the coupled equations (6) and (8). In calculating this distribution, Brederode and Bradshaw's recommended values for the log-law constants[5] were adopted: $\kappa = 0.41$ and $B = 5.2$.

Recently, Hammond[6] and Snijders et al.[7] derived independently an inner layer mean temperature profile expression analogous to Spalding's velocity function [Eq. (6)]. Hammond's expression was originally developed in a rather more complex form applicable to the complete temperature profile within the plane wall-jet. However, in a constant-heat-flux, near-wall layer, his formula reduced to that of Snijders et al.,

$$y^+ = T^+ Pr^{-1} + e^{-A_\theta}\left[e^{\kappa_\theta T^+} - 1 - \sum_{n=1}^{n=4}(\kappa_\theta T^+)^n/n!\right] \tag{9}$$

where $A_\theta = \kappa_\theta B_\theta$. This expression satisfies limiting conditions analogous to those imposed on the velocity profile,

$$T^+ = 0 \text{ and } dT^+/dy^+ = Pr, \text{ at } y^+ = 0 \tag{10}$$

$$T^+ = (1/\kappa_\theta)\ln y^+ + B_\theta(Pr), \text{ for } y^+ Pr > 30\text{-}50 \tag{11}$$

Kader and Yaglom's recommendations for the thermal log-law constants[8] were adopted in the present study. These imply a value for the turbulent Prandtl number of 0.85 at *high-turbulence Reynolds numbers,* where $Pr_t = \kappa/\kappa_\theta$, giving $\kappa_\theta = 0.48$. The molecular Prandtl number dependence of the corresponding "additive constant" is given by[6,8]

$$B_\theta = 12.5\,Pr^{2/3} - 5.8 \tag{12}$$

The "eddy conductivity" within the constant-heat-flux region, where $q^+ = 1$, can be rewritten in the form

$$\epsilon_\theta^+ = (dy^+/dT^+) - (1/Pr) \tag{13}$$

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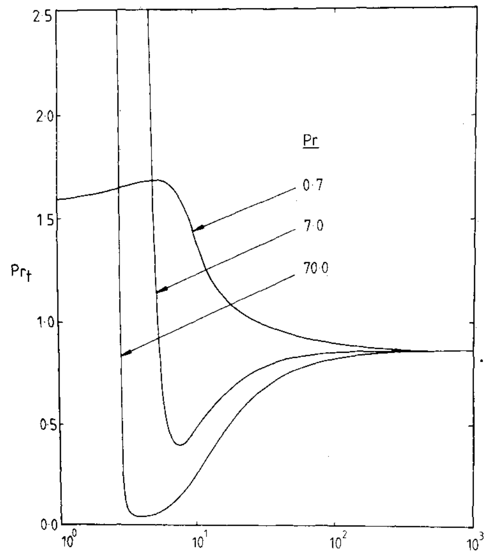

Fig. 1 Near-wall turbulent Prandtl number distribution according to profile analysis.

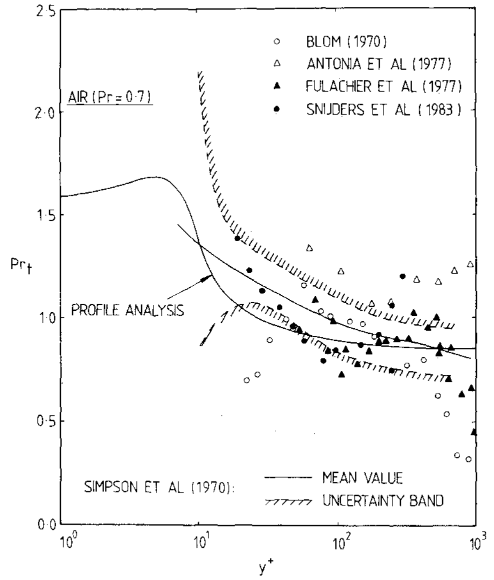

Fig. 2 Turbulent Prandtl number: comparison between profile analysis and experimental data.

via Eq. (2). Its distribution may therefore be obtained by differentiating the temperature profile expression (9) and substituting into Eq. (13) to yield

$$\epsilon_\theta^+ = \kappa_\theta e^{-A_\theta}\left[e^{\kappa_\theta T^+} - 1 - \sum_{n=1}^{n=3} (\kappa_\theta T^+)^n/n!\right] \tag{14}$$

The $\epsilon_\theta^+ - y^+$ profile, computed from the numerical solution of Eqs. (9) and (14) in a similar manner to the eddy viscosity distribution, is shown in the full report[1] for values of $Pr = 0.7$, 7.0, and 70.0. This molecular Prandtl number range covers most of the common, nonmetallic fluids including air, water, and technical oil.[8] The corresponding Pr_t profiles can be readily obtained from Eq. (3), and are presented in Fig. 1. They display a rather exotic molecular Prandtl number dependence in the diffusive sublayer. However, this behavior is simply the consequence of the restrictions placed upon Pr_t by its defining equations (1-3), coupled with the limiting conditions represented in Eqs. (4), (5), (10), and (11).

Experimental Comparison

The reviews by Reynolds[2] and Launder[3] emphasize the sparseness and contradictory nature of the Pr_t measurements in the diffusive sublayer. This is mainly due to the well-known difficulties in making measurements of the Reynolds fluxes and mean property gradients in a small volume close to the wall. Five experimental data sets for air[1] are plotted in Fig. 2, where they are compared with the Pr_t distribution obtained by profile analysis for $Pr = 0.7$. The measurements were made in essentially fully-developed turbulent boundary layers on smooth surfaces, in which the inner layer extends in practice to about $y^+ = 200\text{-}300$. Outside this limit, the influence of the freestream conditions is felt and Pr_t might be expected to differ from its log-law value ($= \kappa/\kappa_\theta$). The present inner layer profile lends support to the data of Simpson et al. (who evaluated Pr_t from the mean properties alone) and Snijders et al.[7] The trend in the measurements of Antonia et al. is also correct, although the values are a little high. Only the early flux-based measurements of Blom display an opposite trend as the wall is approached to that obtained by profile analysis.

Concluding Remarks

Further discussion of the significance of the present results for the use of the turbulent Prandtl number concept can be found in the full report.[1] Tentative suggestions are also made for the treatment of the near-wall region in numerical calculation procedures for thin shear layers. These take the form of recommendations as to which of the many phenomenological models[1-3] show reasonable agreement with the Pr_t distribution implied by profile analysis. The latter is rather complicated and is only implicit in y^+, whereas the former models are generally simpler and explicit in y^+.

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